

ONE FORMULATION FOR BOTH LINEAL GRAVITIES
THROUGH A DIMENSIONAL REDUCTION*

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ABSTRACT

The two lineal gravities — based on the de Sitter group or a central extension of the Poincaré group in $1+1$ dimensions — are shown to derive classically from a unique topological gauge theory. This one is obtained after a dimensional reduction of a Chern–Simons model, which describes pure gravity in $2+1$ dimensions, the gauge symmetry being given by an extension of $ISO(2,1)$.

INTRODUCTION

There is increasing interest these last years in studying gravity in low-dimensional space-times. In $2 + 1$ dimensions the Einstein equations undergo a drastic simplification which allows a deeper understanding of global gravitational effects.¹ Moreover, in the absence of matter, they were shown² to be the equations of motion of a topological Chern–Simons theory. According to the sign of the cosmological constant Λ , the underlying gauge group is usually taken to be $ISO(2, 1)$ for $\Lambda = 0$ (the Poincaré group), $SO(3, 1)$ for $\Lambda > 0$ (the de Sitter group) or $SO(2, 2)$ for $\Lambda < 0$ (the anti-de Sitter group).

In $1 + 1$ dimensions, in spite of the vanishing of the Einstein tensor, two alternatives were proposed as lineal gravities. Both models were shown to be classically — and perhaps also quantically³ — equivalent to a topological gauge theory. Their actions have the general form:

$$L = \int_{\Sigma} \langle H, F \rangle \quad , \quad (1)$$

where F is a (curvature) two-form, H is a Lagrange multiplier scalar function — both take value in a Lie algebra and transform according to the adjoint representation — and $\langle \ , \ \rangle$ defines an invariant non-degenerate bilinear form on this Lie algebra.⁴

In the first case,⁵ the gauge symmetry is given by the (anti-) de Sitter $SO(2, 1)$ group:

$$[P_a, J] = \epsilon_a^{\ b} P_b \quad , \quad [P_a, P_b] = -\frac{\Lambda}{2} \epsilon_{ab} J \quad . \quad (2)$$

Using the inner product coded in the Casimir $P_a P^a + \frac{\Lambda}{2} J^2$ and writing $A = e^a P_a + \omega J$, $H = \eta^a P_a + \frac{\Lambda}{2} \eta_2 J$, the Lagrange density in Eq. (1) is:

$$\mathcal{L} = \eta_a (de^a + \epsilon^a_{\ b} \omega e^b) + \eta_2 \left(d\omega - \frac{1}{4} \Lambda \epsilon_{ab} e^a e^b \right) \quad . \quad (3)$$

[The indices (a, b, \dots) take the value 0,1 and are raised or lowered with the flat-space metric $\eta_{ab} = \text{diag}(1, -1)$. ϵ^{ab} is the anti-symmetric tensor with $\epsilon^{01} = 1$.] The components of the gauge field e^a, ω are interpreted as the *Zweibein* and the spin-connection one-forms. The Lagrange multiplier functions η_a, η_2 enforce the scalar curvature to be equated to the constant Λ . This is called the *de Sitter* model.

On the other hand, the gauge symmetry of the second proposal was recently⁶ identified as a central extension of the Poincaré algebra:

$$[P_a, J] = \epsilon_a{}^b P_b \quad , \quad [P_a, P_b] = \epsilon_{ab} I \quad , \quad [I, J] = 0 = [I, P_a] \quad . \quad (4)$$

Therefore we shall call this model the *extended Poincaré* model. With the inner product given by the Casimir $P_a P^a - JI - IJ$ and the decomposition $A = e^a P_a + \omega J + aI$, $H = \eta^a P_a - \eta_3 J - \eta_2 I$, the Lagrangian density in Eq. (1) becomes:

$$\mathcal{L} = \eta_a (de^a + \epsilon^a{}_{b\omega} e^b) + \eta_2 d\omega + \eta_3 \left(da + \frac{1}{2} \epsilon_{ab} e^a e^b \right) \quad . \quad (5)$$

Remark that now one of the equations of motion set the scalar curvature to zero and that η_3 plays the role of a cosmological constant.⁷ A non-conventional contraction from the de Sitter algebra to the extended Poincaré one relates both models. Namely, adding a $U(1)$ generator I in the algebra (2), replacing J by $J - 2I/\Lambda$ and taking the limit $\Lambda \rightarrow 0$ lead to the algebra (4).

If it was already remarked that the de Sitter model can be viewed as a dimensional reduction of a Chern–Simons model with 2+1-de Sitter gauge group, it was not clear whether the extended Poincaré theory follows from a similar reduction. We show in this paper that this can indeed be achieved provided we start in 2+1 dimensions from an extension of the Poincaré

$ISO(2,1)$ group. We first propose a topological gauge theory based on a new symmetry as pure gravity in $2+1$ dimensions. The dimensionally reduced equations of motion admit then not only the extended Poincaré classical solutions but also the de Sitter ones. We thus get a unified picture of the two different lineal gravities.

In Section I we construct the minimal extension of the $2+1$ -Poincaré algebra which contains the extended $1+1$ -Poincaré one. In Section II we perform a dimensional reduction and we derive the general equations of motion we are left with. They admit Einstein-type solutions with an arbitrary cosmological constant. In Section III we show that the de Sitter and extended Poincaré classical solutions are among them. Finally we give some comments and conclusions in Section IV.

I. THE MINIMAL EXTENSION OF $ISO(2,1)$ AND $2+1$ -GRAVITY

It is always interesting to look at a theory as the dimensional reduction of another one. The extra space-time dimensions carry additional information describing, for example, electromagnetism in the Kaluza-Klein model⁸ or a Higgs field in the Manton–Meyer model.⁹ In our case we look for gravity theories in $2+1$ dimensions which reduce in $1+1$ dimensions to the two lineal gravities we have just described. The dimensional reduction that will be explicitly shown in the next section consists of imposing translational invariance along a spatial direction. Contrary to the cited cases, this reduction does not change the gauge symmetry and hence the gauge group is not modified.

The last remark has lead us to search for a group containing the extended Poincaré one (4) and which still gives a description of gravity in $2+1$ dimensions. This can be achieved by

an extension of $ISO(2, 1)$ similar to the one of $ISO(1, 1)$. We notice that a central extension of $ISO(2, 1)$ does not exist. Let us consider the (anti) de Sitter algebra in $2 + 1$ dimensions:

$$[\bar{J}_A, \bar{J}_B] = \epsilon_{AB}^{C} \bar{J}_C \quad , \quad [\bar{J}_A, P_B] = \epsilon_{AB}^{C} P_C \quad , \quad [P_A, P_B] = -\Lambda \epsilon_{AB}^{C} \bar{J}_C \quad (6)$$

[the indices (A, B, C, \dots) take the value $0, 1, 2$ and are moved with the metric $\eta_{AB} = \text{diag}(1, -1, -1)$. ϵ^{ABC} is the totally antisymmetric tensor with $\epsilon^{012} = 1$]. We extend it trivially by $SU(2)$:

$$[S_A, S_B] = \epsilon_{AB}^{C} S_C \quad , \quad [S_A, \bar{J}_B] = 0 \quad , \quad [S_A, P_B] = 0 \quad (7)$$

and we perform the contraction, $J_A = \bar{J}_A + S_A$, $I_A = -\Lambda S_A$, $\Lambda \rightarrow 0$:

$$\begin{aligned} [J_A, J_B] &= \epsilon_{AB}^{C} J_C \quad , \\ [J_A, P_B] &= \epsilon_{AB}^{C} P_C \quad , \quad [P_A, P_B] = \epsilon_{AB}^{C} I_C \quad , \\ [J_A, I_B] &= \epsilon_{AB}^{C} I_C \quad , \quad [P_A, I_B] = 0 \quad , \quad [I_A, I_B] = 0 \quad . \end{aligned} \quad (8)$$

This algebra is an extension of the Poincaré algebra by an Abelian ideal (of dimension three).

To write the Chern–Simons action with the gauge symmetry (8) we need an invariant non-degenerate bilinear form. The most general one is easily shown to be parametrized by two real constants c_1, c_2 . In the basis $\{T_M\}_{M=1}^9 = \{P_A, J_A, I_A\}_{A=0}^2$ we get:

$$\langle T_M, T_N \rangle \equiv h_{MN} = \begin{pmatrix} \eta_{AB} & c_2 \eta_{AB} & 0 \\ c_2 \eta_{AB} & c_1 \eta_{AB} & \eta_{AB} \\ 0 & \eta_{AB} & 0 \end{pmatrix} \quad (9)$$

which is associated with the Casimir:

$$C = P_A P^A - c_2 (P_A I^A + I_A P^A) + (J_A I^A + I_A J^A) + (c_2^2 - c_1) I_A I^A \quad . \quad (10)$$

With $\tilde{A} = e^A P_A + \omega^A J_A + a^A I_A$ a Lie algebra valued one-form, our action is written:

$$\begin{aligned}\tilde{L} &= \int_{\mathbb{R}^3} \left\langle \tilde{A}, d\tilde{A} + \frac{2}{3}\tilde{A}^2 \right\rangle = \int d^3x \tilde{\mathcal{L}} \\ \tilde{\mathcal{L}} &= 2c_2 e_A \left(d\omega^A + \frac{1}{2}\epsilon^A_{BC}\omega^B\omega^C \right) + e_A (de^A + \epsilon^A_{BC}\omega^B e^C) \\ &\quad + 2a_A \left(d\omega^A + \frac{1}{2}\epsilon^A_{BC}\omega^B\omega^C \right) + c_1 \left(\omega_A d\omega^A + \frac{1}{3}\epsilon_{ABC}\omega^A\omega^B\omega^C \right) .\end{aligned}\tag{11}$$

(We have dropped in $\tilde{\mathcal{L}}$ some exact differentials contributing by surface terms only.) If e^A, ω^A are interpreted as a *Dreibein* and a spin-connection, the first term is just the scalar curvature and the second one the scalar torsion.¹⁰

The relation with pure gravity is still clearer if we present the equations of motion derived from the action (11). For arbitrary c_1, c_2 , the equations of motion are always given by the zero curvature condition, $d\tilde{A} + \tilde{A}^2 = 0$; in components we get:

$$\begin{aligned}de^A + \epsilon^A_{BC}\omega^B e^C &= 0 \quad , \\ d\omega^A + \frac{1}{2}\epsilon^A_{BC}\omega^B\omega^C &= 0 \quad , \\ da^A + \epsilon^A_{BC}\omega^B a^C + \frac{1}{2}\epsilon^A_{BC}e^B e^C &= 0 \quad .\end{aligned}\tag{12}$$

We recognize in the two first equations the torsion free and the Einstein equations of pure gravity without cosmological constant. The role of the field a^A is still to be elucidated. If it appears only as an auxiliary field at the classical level, its presence in the Lagrangian (11) could have a decisive contribution in the quantization procedure and induce different results than the usual model.² Thus Eq. (11) is an alternative to pure gravity (without cosmological constant) based on an extension of $ISO(2, 1)$.

II. THE DIMENSIONALLY REDUCED MODEL

We now descend to $1 + 1$ dimensions by imposing a translational invariance along the second spatial direction. It is useful to write the one-form $\tilde{A} = A + A_2 dx^2$ and to introduce

the curvature two-form $F = dA + A^2$. Apart from a surface term and an (infinite) constant generated by the integration along x^2 , the action (11) is reduced to:

$$L = \int_{\mathbb{R}^2} \langle A_2, F \rangle \quad . \quad (13)$$

If we identify A_2 with the scalar function H , we recognize the topological action (1) in 1 + 1 dimensions. Using the inner product (9) in Eq. (13) we can rearrange the terms with the help of the convenient definitions:

$$A_{2,M} \equiv h_{MN} A_2^N \equiv \left(\left(\eta_a^{(1)}, \eta^{(1)} \right), \left(\eta_a^{(2)}, \eta^{(2)} \right), \left(\eta_a^{(3)}, \eta^{(3)} \right) \right) \quad (14)$$

[we recall that (a, b, \dots) takes the value 0,1 only and are moved with the metric $\text{diag}(1, -1)$].

Due to the invertibility of the inner product, these fields are independent. If the reduced connection A ($\tilde{A} = A + A_2 dx^2$) is decomposed according to:

$$A = e^a P_a + e P_2 + \omega^a J_a + \omega J_2 + a^a I_a + a I_2 \quad (15)$$

the Lagrangian density in (13) becomes:

$$\begin{aligned} \mathcal{L} = & \eta_a^{(1)} (de^a + \epsilon^a_b \omega e^b + \epsilon^a_b e \omega^b) + \eta_a^{(2)} (d\omega^a + \epsilon^a_b \omega \omega^b) \\ & + \eta_a^{(3)} (da^a + \epsilon^a_b \omega a^b + \epsilon^a_b e e^b + \epsilon^a_b a \omega^b) \\ & + \eta^{(1)} (de + \epsilon_{ab} \omega^a e^b) + \eta^{(2)} \left(d\omega + \frac{1}{2} \epsilon_{ab} \omega^a \omega^b \right) \\ & + \eta^{(3)} \left(da + \frac{1}{2} \epsilon_{ab} e^a e^b + \epsilon_{ab} \omega^a a^b \right) \quad , \end{aligned} \quad (16)$$

and will be called the *reduced* Lagrangian.

Interpreting e^a as a *Zweibein* and ω as a spin-connection, we recognize the torsion tensor $de^a + \epsilon^a_b \omega e^b$ multiplying $\eta_a^{(1)}$ and the scalar curvature $d\omega$ multiplying $\eta^{(2)}$. For a theory of gravitation we want the *Zweibein* and the spin connection to be related — at least classically

— by a torsion free condition. This is obtained by varying (16) with respect to $\eta_a^{(1)}$ if and only if either e or ω^a is identically zero.

Let us look first if such a choice is consistent with the other equations of motion. The ones involved are obtained by variations with respect to $\eta^{(1)}$ and $\eta_a^{(2)}$:

$$de + \epsilon_{ab}\omega^a e^b = 0 \quad (17a)$$

$$d\omega^a + \epsilon^a{}_b \omega \omega^b = 0 \quad . \quad (17b)$$

Try first $e \equiv 0$. Then (17a) implies for ω^a :

$$\omega^a = \sqrt{\frac{\Lambda}{2}} \epsilon^a{}_b e^b \quad (18)$$

with an arbitrary positive constant Λ (possibly set to zero). Then (17b) gives just a multiple of the torsion free condition and $e \equiv 0$ is a consistent choice

By a variation with respect to $\eta_a^{(1)}$ and $\eta^{(2)}$ and using the previous solution for e and ω^a we get:

$$de^a + \epsilon^a{}_b \omega e^b = 0 \quad , \quad d\omega - \frac{1}{4} \Lambda \epsilon_{ab} e^a e^b = 0 \quad (19)$$

which are the basic equations of the de Sitter model if $\Lambda \neq 0$ and of the extended Poincaré model if $\Lambda \equiv 0$. The choice $\omega^a = 0$ is also consistent and implies e to be constant [cf. Eq. (17a)]. Equation (19) holds with $\Lambda = 0$.

We conclude this section by emphasizing the result (19). Einstein-type gravities with arbitrary cosmological constant are among the solutions of the reduced equations of motion [coming from Lagrangian (16)]. They are characterized by a vanishing torsion tensor. Another intriguing point is the gauge structure of Lagrangian (16). After the reduction, the symmetry is still given by the extended 2 + 1-Poincaré algebra. Even if this seems not to be a “natural” algebra in 1 + 1 dimensions, the extended Poincaré *and* the de Sitter gauge structures can be recovered on-shell as we shall show in the next section.

III. THE EXTENDED POINCARÉ AND THE DE SITTER MODELS IN LIGHT OF THE REDUCED MODEL

We present here how the classical solutions of the extended Poincaré and the de Sitter actions lie among the classical solutions of the reduced action. More precisely, we show how special *Ansätze* reduce the equations coming from (16) to those coming from (5) or (3).

The extended 1 + 1-Poincaré algebra (4) has a natural embedding in the extended 2 + 1-Poincaré algebra (8). One checks that the generators:

$$P'_a = P_A \quad , \quad J' = J_2 \quad , \quad I' = I_2 \quad (20)$$

span in the algebra (8) a subalgebra which is exactly the one given by Eq. (4). On the other hand, the linear combinations

$$\begin{aligned} P'_a &= P_a - \sqrt{\frac{\Lambda}{2}} \epsilon_a{}^b J_b - \frac{1}{\sqrt{2\Lambda}} \epsilon_a{}^b I_b \quad , \\ J' &= J_2 \end{aligned} \quad (21)$$

generate the de Sitter algebra (2) in 1 + 1 dimensions. In other words we notice that the extended 2 + 1-Poincaré algebra contains both the extended Poincaré and de Sitter algebras.

Moreover, the inner product (9) induces in these subalgebras the general invariant inner products of the de Sitter and the extended Poincaré algebras. This means that restricting the gauge field in Eq. (16) to one of the algebras (20) or (21) we get the usual Lagrangian densities (3) or (5). Their classical solutions are thus among the ones of the reduced model.

More explicitly, the *Ansatz*:

$$\begin{aligned} A &= e^a \left(P_a - \sqrt{\frac{\Lambda}{2}} \epsilon_a{}^b J_b - \frac{1}{\sqrt{2\Lambda}} \epsilon_a{}^b I_b \right) + \omega J_2 \quad , \\ H &= \eta^a \left(P_a - \sqrt{\frac{\Lambda}{2}} \epsilon_a{}^b J_b - \frac{1}{\sqrt{2\Lambda}} \epsilon_a{}^b I_b \right) + \frac{\Lambda}{2} \eta_2 J_2 \quad , \end{aligned} \quad (22)$$

allows us to solve the equations of motion of the reduced model and the solutions coincide with the de Sitter ones if e^a , ω , η^a , η_2 correspond to the fields in (3). The anti-de Sitter model (with Λ negative) can be treated in a similar way, but we do not present the details here. The same remark applies to the extended Poincaré model with the *Ansatz*:

$$\begin{aligned} A &= e^a P_A + \omega J_2 + a I_2 \quad , \\ H &= \eta^a P_A + \eta_3 J_2 + \eta_2 I_2 \quad . \end{aligned} \tag{23}$$

Here e^a , ω , a , η^a , η_2 , η_3 are identified with the corresponding fields in (5).

IV. COMMENTS AND CONCLUSIONS

We have shown that the equations of motion of the reduced model (16) possesses both the de Sitter ($\Lambda \neq 0$) and the extended Poincaré ($\Lambda = 0$) solutions. If we consider a trivial bundle on $\Sigma = \mathbb{R}^2$, the space of classical solutions of the topological model (1) is given by all the flat connections. This space is contractible and if we divide it by all the gauge transformations it reduces to a point. In order to interpret (16) as a model of gravity, we identify in the decomposition (15) e^a as a *Zweibein* and construct a metric in the usual way $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$. But we have to impose that $-\frac{1}{2}\epsilon_{ab}e^ae^b$ never vanishes since it is proportional to $\sqrt{-\det g}$. This means that not all configurations are geometrical solutions of the equations of motion. The configurations with non-vanishing $-\frac{1}{2}\epsilon_{ab}e^ae^b$ form a set which no longer has a trivial topology. For example, in our reduced model, the de Sitter and the extended Poincaré solutions are clearly disconnected.¹¹

In this paper we have obtained the two lineal gravities as a reduction of a 2+1-dimensional topological theory. This one is in fact a model of 2+1-gravity whose symmetry is an extension of the Poincaré algebra. The reduction to 1 + 1-dimensions gives an action whose classical solutions can be the de Sitter or the extended Poincaré ones. The quantization of this model will deserve further study.

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